4501. The relevant expansions are

$$\begin{aligned}z^4 &= (2x-1)^4\\ &\equiv 16x^4 - 32x^3 + 24x^2 - 8x + 1,\\ z^2 &= (2x-1)^2\\ &\equiv 4x^2 - 4x + 1.\end{aligned}$$

Hence, we can write the equation as

$$\begin{aligned}&z^4 + 2z^2 - 24 = 0\\ \implies &(z^2+6)(z^2-4) = 0.\end{aligned}$$

The first factor has no real roots, so $z = \pm 2$. This gives $2x - 1 = \pm 2$, so $x = -\frac{1}{2}, \frac{3}{2}$.

4502. (a) If the iteration has a fixed point α, then

$$\begin{aligned}&\alpha = \alpha - \frac{f(\alpha)}{f'(\alpha)}\\ \implies &\frac{f(\alpha)}{f'(\alpha)} = 0\\ \implies &f(\alpha) = 0.\end{aligned}$$

So, $x = \alpha$ is a root of $f(x) = 0$.

(b) For $x^k = 0$, the iteration is

$$\begin{aligned}x_{n+1} &= x_n - \frac{x_n^k}{kx_n^{n-1}}\\ &\equiv x_n - \frac{x_n}{k}\\ &\equiv x_n\big(1 - \tfrac{1}{k}\big).\end{aligned}$$

This is a GP. The nth term is

$$x_n = x_0\big(1 - \tfrac{1}{k}\big)^n.$$

Since $k > 1$, we know that the common ratio $r = 1 - \frac{1}{k} \in (0, 1)$. So, the GP converges to zero, whatever the value of x_0. □

4503. Separating the variables,

$$3\int \sqrt{y}\,dy = \int \sin^2 x\,dx.$$

Using a double-angle formula, the x integrand is $\frac{1}{2}(1 - \cos 2x)$. This gives

$$\begin{aligned}&3\int \sqrt{y}\,dy = \tfrac{1}{2}\int 1 - \cos 2x\,dx\\ \implies &2y^{\frac{3}{2}} = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c.\end{aligned}$$

Substituting $(0, 0)$, we get $c = 0$. So,

$$\begin{aligned}&y^{\frac{3}{2}} = \tfrac{1}{4}x - \tfrac{1}{8}\sin 2x\\ \implies &y = \big(\tfrac{1}{4}x - \tfrac{1}{8}\sin 2x\big)^{\frac{2}{3}}\\ &\equiv \tfrac{1}{4}\big(2x - \sin 2x\big)^{\frac{2}{3}}, \text{ as required.}\end{aligned}$$

4504. Completing the square,

$$\begin{aligned}y &= x^2 + 2ax + a^2 + b,\\ &\equiv (x+a)^2 + b.\end{aligned}$$

The vertex is at (a, b). So, the image is a monic parabola of the form $x = f(y)$, with a vertex at (a, b). This is

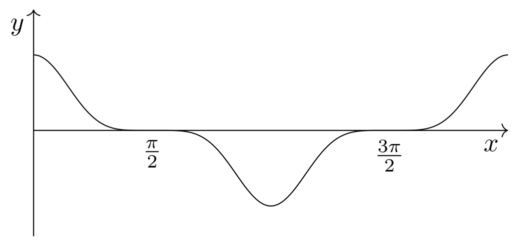
$$\begin{aligned}x &= (y-b)^2 + a\\ &\equiv y^2 - 2by + b^2 + a.\end{aligned}$$

4505. Factorising top and bottom,

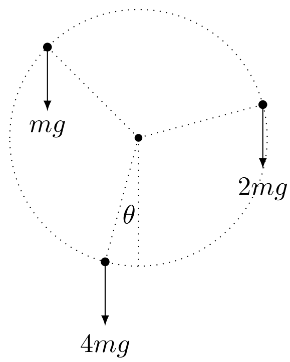
$$\begin{aligned}&\lim_{x\to 1}\frac{x^5-1}{x^5-x}\\ &= \lim_{x\to 1}\frac{(x-1)(x^4+x^3+x^2+x+1)}{x(x^2+1)(x+1)(x-1)}\\ &= \lim_{x\to 1}\frac{x^4+x^3+x^2+x+1}{x(x^2+1)(x+1)}.\end{aligned}$$

We can now take the limit, giving $\frac{5}{4}$.

4506. The effect of raising $\cos x$ to the fifth power is to convert the x axis intercepts of $y = \cos x$ into points of inflection (quintuple roots). The minima and maxima remain the same, as do the positives and negatives. Over the domain $[0, 2\pi)$, this gives



4507. The force diagram is



Call the radius 1. We take anticlockwise moments around the axle, noting that the directions of the moments are taken care of by the ± values of the sine function:

$$\begin{aligned}&4mg\sin\theta + mg\sin\big(\theta + \tfrac{2\pi}{3}\big)\\ &\qquad + 2mg\sin\big(\theta - \tfrac{2\pi}{3}\big) = 0\end{aligned}$$

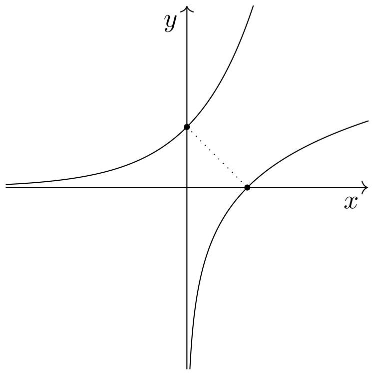
Using compound-angle formulae,

$$\begin{aligned}&\sin\left(\theta \pm \tfrac{2\pi}{3}\right)\\ \equiv\ &\sin\theta\cos\tfrac{2\pi}{3} \pm \cos\theta\sin\tfrac{2\pi}{3}\\ \equiv\ &-\tfrac{1}{2}\sin\theta \pm \tfrac{\sqrt{3}}{2}\cos\theta.\end{aligned}$$

So, the moments equation is

$$\begin{aligned}&4mg\sin\theta + mg\big(-\tfrac{1}{2}\sin\theta + \tfrac{\sqrt{3}}{2}\cos\theta\big)\\ &\quad + 2mg\big(-\tfrac{1}{2}\sin\theta - \tfrac{\sqrt{3}}{2}\cos\theta\big) = 0\\ \Longrightarrow\ &\tfrac{5}{2}mg\sin\theta = \tfrac{\sqrt{3}}{2}mg\cos\theta\\ \Longrightarrow\ &\tan\theta = \tfrac{\sqrt{3}}{5}.\end{aligned}$$

4508. The graphs are reflections in the line $y = x$. Hence, the shortest distance between them must lie along a line of gradient $m = -1$. So, the endpoints of the shortest path are at points with gradient 1. Hence, the relevant points are $(0, 1)$ and $(1, 0)$.



This gives the distance as $\sqrt{2}$.

4509. The integrand simplifies as follows:

$$\begin{aligned}&\frac{225x^2 + 30x + 1}{15x^2 + 31x + 2}\\ \equiv\ &\frac{15(15x^2 + 31x + 2) - 435x - 29}{15x^2 + 31x + 2}\\ \equiv\ &15 - \frac{29(15x + 1)}{(15x + 1)(x + 2)}\\ \equiv\ &15 - \frac{29}{x + 2}.\end{aligned}$$

So, the integral is

$$\begin{aligned}&\int_0^2 15 - \frac{29}{x+2}\,dx\\ =\ &\Big[15x - 29\ln|x + 2|\Big]_0^2\\ =\ &(30 - 29\ln 4) - (-29\ln 2)\\ =\ &30 - 29\ln 2, \text{ as required.}\end{aligned}$$

4510. Let $y = \dfrac{2}{1 - \sqrt{x}}$.

The first principles formula gives

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\frac{2}{1-\sqrt{x+h}} - \frac{2}{1-\sqrt{x}}}{h}.$$

Multiplying top and bottom of the main fraction by the denominators of the inlaid fractions, this is

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{2 - 2\sqrt{x} - \big(2 - 2\sqrt{x+h}\big)}{h\big(1 - \sqrt{x}\big)\big(1 - \sqrt{x+h}\big)}\\ &\equiv \lim_{h\to 0}\frac{2\big(\sqrt{x+h} - \sqrt{x}\big)}{h\big(1 - \sqrt{x}\big)\big(1 - \sqrt{x+h}\big)}.\end{aligned}$$

Call the conjugate $C = \sqrt{x+h} + \sqrt{x}$. Multiplying top and bottom by C, the numerator becomes a difference of two squares:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{2\big((x+h) - x\big)}{hC\big(1 - \sqrt{x}\big)\big(1 - \sqrt{x+h}\big)}\\ &\equiv \lim_{h\to 0}\frac{2h}{hC\big(1 - \sqrt{x}\big)\big(1 - \sqrt{x+h}\big)}\\ &\equiv \lim_{h\to 0}\frac{2}{C\big(1 - \sqrt{x}\big)\big(1 - \sqrt{x+h}\big)}.\end{aligned}$$

At this point, we can take the limit. The RH factor on the bottom tends to $\big(1 - \sqrt{x}\big)$ and C tends to $2\sqrt{x}$. This gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{2\sqrt{x}\big(1 - \sqrt{x}\big)^2}\\ &\equiv \frac{1}{\sqrt{x}\big(1 - \sqrt{x}\big)^2}, \text{ as required.}\end{aligned}$$

4511. (a) Consider $x \geq 0$. The equation is

$$(y - x)^2 + y^2 = 1.$$

Differentiating implicitly with respect to x,

$$2(y - x)\big(\tfrac{dy}{dx} - 1\big) + 2y\tfrac{dy}{dx} = 0.$$

Setting $\frac{dy}{dx} = 0$, this is $y = x$. Substituting this in, we get $y^2 = 1$, so $y = \pm 1$.

- The value $y = 1$ gives a tangent parallel to the x axis at $(1, 1)$.
- The value $y = -1$ gives $(-1, -1)$. Since we are considering only $x \geq 0$, however, this is a phantom solution.

Mirroring the point $(1, 1)$ in $x = 0$, there are tangents parallel to the x axis at $(\pm 1, 1)$.

(b) Again, let $x \geq 0$. This time, we differentiate implicitly with respect to y,

$$2(y - x)\big(1 - \tfrac{dx}{dy}\big) + 2y = 0.$$

Setting $\frac{dx}{dy} = 0$, this is $2y - 2x + 2y = 0$, so $2y = x$. Substituting in,

$$\begin{aligned}&(y - 2y)^2 + y^2 = 1\\ \Longrightarrow\ &y^2 = \tfrac{1}{2}\\ \Longrightarrow\ &y = \pm\tfrac{\sqrt{2}}{2}.\end{aligned}$$

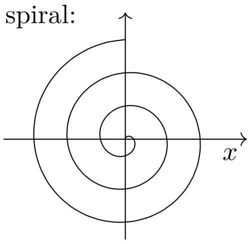
Again, negative y produces negative x, which we reject. Positive y gives $\big(\sqrt{2}, \sqrt{2}/2\big)$. We then mirror this point in $x = 0$, to give tangents parallel to the y axis at $\big(\pm\sqrt{2}, \sqrt{2}/2\big)$.

4512. (a) The distance r is given by

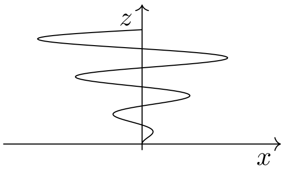
$$\begin{aligned} r &= \sqrt{t^2 \sin^2 t + t^2 \cos^2 t + t^2} \\ &\equiv \sqrt{2t^2}. \end{aligned}$$

Since $t \geq 0$, $r = \sqrt{2}t$. So, $\frac{dr}{dt} = \sqrt{2}$, meaning that the distance from the origin is increasing at a linear rate.

(b) In the (x, y) plane, we have $x = t \sin t$ and $y = t \cos t$. This is circular motion, combined with a constant scale factor of enlargement, producing a spiral:



In the (x, z) plane, we have $x = t \sin t$ and $z = t$. This is linear motion in the z direction, combined with an oscillation in x of linearly increasing amplitude:



Overall, the motion is an expanding circular spiral in the horizontal (x, y) plane, combined with a constant rate of increase of altitude z.

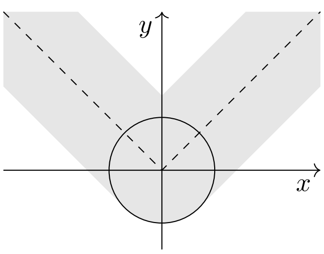
4513. Let $\frac{1}{2}\theta = \phi$. The RHS is then

$$\begin{aligned} &\frac{1 - \cos\theta}{\sin\theta} \\ \equiv\ &\frac{1 - \cos 2\phi}{\sin 2\phi}. \end{aligned}$$

Using double-angle formulae, this is

$$\begin{aligned} &\frac{2\sin^2\phi}{2\sin\phi\cos\phi} \\ \equiv\ &\frac{\sin\phi}{\cos\phi} \\ \equiv\ &\tan\phi \\ \equiv\ &\tan\tfrac{1}{2}\theta, \text{ as required.} \end{aligned}$$

4514. The circles have radius 1, and are centred at $x = p$, $y = |p|$. This has Cartesian equation $y = |x|$. For a point to lie on at least one of the circles, it must lie within 1 unit distance of $y = |x|$:



4515. Substituting for z in the first equation,

$$\begin{aligned} & \ln x + \ln y = 2\ln\left(\tfrac{1}{2}(x + y)\right) \\ \Longrightarrow\ & \ln xy = \ln\left(\tfrac{1}{2}(x + y)\right)^2 \\ \Longrightarrow\ & xy = \left(\tfrac{1}{2}(x + y)\right)^2 \\ \Longrightarrow\ & 0 = x^2 - 2xy + y^2 \\ \Longrightarrow\ & 0 = (x - y)^2 \\ \Longrightarrow\ & x = y, \text{ as required.} \end{aligned}$$

4516. (a) Let $\tan y = x$. Differentiating implicitly,

$$\begin{aligned} & \sec^2 y \frac{dy}{dx} = 1 \\ \Longrightarrow\ & (1 + \tan^2 y)\frac{dy}{dx} = 1 \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{1}{1 + \tan^2 y} \\ & \qquad = \frac{1}{1 + x^2}. \end{aligned}$$

(b) Let $u = \arctan x$ and $v' = 1$. So, $v = x$ and

$$u' = \frac{1}{1 + x^2}.$$

Using the parts formula,

$$\begin{aligned} & \int \arctan x \, dx \\ &= x \arctan x - \int \frac{x}{1 + x^2} \, dx \\ &= x \arctan x - \tfrac{1}{2} \int \frac{2x}{1 + x^2} \, dx \\ &= x \arctan x - \tfrac{1}{2} \ln\left(1 + x^2\right) + c. \end{aligned}$$

4517. Consider a tangent line with equation $y = \mathrm{f}(x)$. This intersects the curve where

$$\begin{aligned} & x^3 - x = \mathrm{f}(x) \\ \Longleftrightarrow\ & x^3 - x - \mathrm{f}(x) = 0. \end{aligned}$$

This is a cubic equation. We know that it has a repeated root at the point of tangency. This must be a double root or a triple root.

- If it is a double root (squared factor), then factorising leaves a linear factor. This gives a re-intersection point.
- If it is a triple root (cubed factor), then the above argument doesn't apply. This occurs with the tangent line $y = x$ at $x = 0$, which intersects the curve exactly once.

Therefore, the origin is the only point at which a tangent can be drawn that doesn't re-intersect the curve. □

4518. Differentiating with respect to y,

$$x = \left(1 - y^2\right)^{\frac{3}{2}}$$
$$\Longrightarrow \frac{dx}{dy} = -3y\left(1 - y^2\right)^{\frac{1}{2}}.$$

The normal gradient is the negative reciprocal of ${}^{dy}/_{dx}$, which is the negative of ${}^{dx}/_{dy}$. Therefore,

$$m = 3y\left(1 - y^2\right)^{\frac{1}{2}}.$$

From the origin to the point (x, y), the gradient is ${}^{y}/_{x}$. We need this to be equal to m. This gives

$$\frac{y}{\left(1 - y^2\right)^{\frac{3}{2}}} = 3y\left(1 - y^2\right)^{\frac{1}{2}}$$
$$\Longrightarrow y = 3y\left(1 - y^2\right)^2.$$

We reject the root $y = 0$. This gives

$$1 = 3\left(1 - y^2\right)^2$$
$$\Longrightarrow 3y^4 - 6y^2 + 2 = 0$$
$$\Longrightarrow y = \pm 1.25593, \pm 0.650115.$$

We reject $y = \pm 1.25593$: there are no such points on the curve. The relevant value is $y = 0.650115$. We substitute this back into

$$m = 3y\left(1 - y^2\right)^{\frac{1}{2}}.$$

The gradient of the normal is 1.48 (3sf).

4519. Call the colours RGBY. The orders of RGB are

RGB	BRG	GBR
RBG	BGR	GRB

These all constitute the same bracelet: rotations along the circumference cycle across the rows, and rotations around a diameter cycle down the columns. Choose RGB without loss of generality. The only choice is where to put Y. There are three positions to choose from, so three arrangements of the beads:

RGBY   RGYB   RYGB.

— Nota Bene —

It might seem that there are *four* positions for Y, of which only three are shown above. But YRGB does not constitute a new arrangement, as it is identical to RGBY.

4520. The DE is a quadratic in $\frac{dy}{dx}$. Factorising,

$$\left(\frac{dy}{dx} + 4\right)\left(\frac{dy}{dx} - 3\right) = 0$$
$$\Longrightarrow \frac{dy}{dx} = -4 \text{ or } 3.$$

The gradient of a polynomial $y = \mathrm{f}(x)$ cannot jump from -4 to 3, so it must be one or the other. The possible solutions are $y = -4x + c$ or $y = 3x + d$.

4521. We solve by factorising:

$$\cos^3\theta \sin\theta - 2\sin\theta + \tfrac{1}{2}\cos^3\theta - 1 = 0$$
$$\Longrightarrow \left(\cos^3\theta - 2\right)\left(\sin\theta + \tfrac{1}{2}\right) = 0.$$

The first factor gives no roots, as the range of $\cos^3\theta$ is $[-1, 1]$. This leaves $\sin\theta = -{}^1/_2$, so the solution is $\theta = -{}^{\pi}/_6, -{}^{5\pi}/_6$.

4522. Placing the perpendicular sides of the triangle on the x and y axes, the equation of the hypotenuse is $y = 3 - \frac{3}{4}x$. The centre of the circle is then (k, k), and it has equation

$$(x - k)^2 + (y - k)^2 = k^2.$$

We need this to have exactly one intersection with the hypotenuse. Substituting the equation of the hypotenuse,

$$(x - k)^2 + \left(3 - \tfrac{3}{4}x - k\right)^2 = k^2$$
$$\Longrightarrow 25x^2 - (8k + 72)x + 144 - 96k + 16k^2 = 0.$$

Setting the discriminant to zero,

$$(8k + 72)^2 - 100(144 - 96k + 16k^2) = 0$$
$$\Longrightarrow k = 1, 6.$$

We reject $k = 6$, which is outside the triangle. Hence, the largest circle which fits inside a $(3, 4, 5)$ triangle has radius 1.

4523. (a) As x and y get very large, the 1 becomes negligible next to x^3 and y^3. Hence, C tends towards $y^3 = x^3$, which is $y = x$.

(b) Differentiating with respect to x,

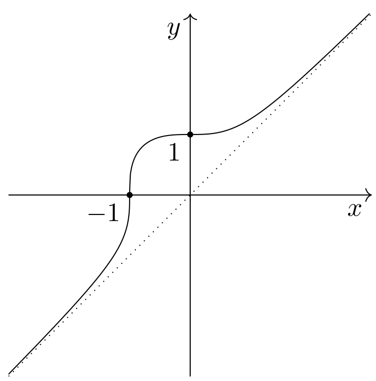
$$3y^2\frac{dy}{dx} = 3x^2.$$

Setting $\frac{dy}{dx} = 0$ gives $x = 0$. So, the curve is parallel to the x axis at $(0, 1)$.
Differentiating with respect to y,

$$3y^2 = 3x^2\frac{dx}{dy}.$$

Setting $\frac{dx}{dy} = 0$ gives $y = 0$. So, the curve is parallel to the y axis at $(-1, 0)$.

(c) Putting the above information together, C is



4524. The quartic is invertible over two intervals whose union is $\mathbb{R}$. So, since a quartic must have a turning point, $\big(p, \mathrm{f}(p)\big)$ must be that turning point. There can be no others. Also, the quartic is monic, so it is positive (leading coefficient 1). Furthermore, we are told that $\mathrm{f}(p) > 0$, which means that the global minimum of the quartic is positive. Hence, $y = \mathrm{f}(x)$ is above the x axis everywhere, and the equation $\mathrm{f}(x) = 0$ can have no real roots.

4525. The problem has two stages. The first stage is a pulley system with masses $6m$ and $4m$. This lasts for one second. The second stage is freefall and a pulley system with masses $6m$ and $2m$.

① The equation of motion along the string is

$$\begin{aligned} 6mg - 4mg &= 6ma + 4ma \\ \Longrightarrow a &= \tfrac{1}{5}g. \end{aligned}$$

So, after one second, each monkey is moving at $\frac{1}{5}g$ ms^{-1}. The vertical distance between the two monkeys is

$$\begin{aligned} s &= 2\big(ut + \tfrac{1}{2}at^2\big) \\ &= \tfrac{1}{5}g. \end{aligned}$$

② In the second stage, NII for the system is

$$\begin{aligned} 6mg - 2mg &= 6ma + 2ma \\ \Longrightarrow a &= \tfrac{1}{2}g. \end{aligned}$$

The victim is in freefall. So, relative to the thief, the conditions are $h_0 = \frac{1}{5}g$, $u = \frac{2}{5}g$, $a = -\frac{1}{2}g$. This gives

$$\begin{aligned} -\tfrac{1}{5}g &= \tfrac{2}{5}gt - \tfrac{1}{4}gt^2 \\ \Longrightarrow 5t^2 - 8t - 4 &= 0 \\ \Longrightarrow t &= -\tfrac{2}{5}, 2. \end{aligned}$$

The negative root isn't relevant. The victim catches the thief after 2 seconds.

4526. Since $x \equiv e^{\ln x}$, we can write x^x as

$$\begin{aligned} x^x &\equiv (e^{\ln x})^x \\ &\equiv e^{x \ln x}. \end{aligned}$$

By the chain and product rules,

$$\frac{d}{dx}\big(e^{x\ln x}\big) \equiv e^{x \ln x}(\ln x + 1).$$

Differentiating again,

$$\begin{aligned} \frac{d^2}{dx^2}\big(e^{x\ln x}\big) &\equiv e^{x\ln x}(\ln x + 1)^2 + e^{x \ln x} \cdot \frac{1}{x} \\ &\equiv x^x(\ln x + 1)^2 + x^x \cdot x^{-1} \\ &\equiv x^x(\ln x + 1)^2 + x^{x-1}. \end{aligned}$$

4527. The difference $y_1 - y_2$ is given by

$$\begin{aligned} y_1 - y_2 &= x^4 - x^2 - k(x^2 - 1) \\ &\equiv x^4 - (1+k)x^2 + k. \end{aligned}$$

Setting this to zero, we have a quadratic in x^2. We use the quadratic formula:

$$\begin{aligned} x^4 - (1+k)x^2 + k &= 0 \\ \Longrightarrow x^2 &= \frac{1 + k \pm \sqrt{(1+k)^2 - 4k}}{2} \\ &\equiv \frac{1 + k \pm \sqrt{1 - 2k + k^2}}{2} \\ &\equiv \frac{1 + k \pm \sqrt{(1-k)^2}}{2} \\ &\equiv 1, k. \end{aligned}$$

Since $k \in (0, 1)$, this gives four roots

$$x = \pm 1, \pm\sqrt{k}.$$

Half the area of the central region is

$$\begin{aligned} &\int_0^{\sqrt{k}} x^4 - (1+k)x^2 + k \, dx \\ &= \Big[\tfrac{1}{5}x^5 - \tfrac{1}{3}(1+k)x^3 + kx\Big]_0^{\sqrt{k}} \\ &\equiv \tfrac{1}{5}k^{\frac{5}{2}} - \tfrac{1}{3}(1+k)k^{\frac{3}{2}} + k \cdot k^{\frac{1}{2}} \\ &\equiv -\tfrac{2}{15}k^{\frac{5}{2}} + \tfrac{2}{3}k^{\frac{3}{2}}. \end{aligned}$$

The area of the right-hand region is

$$\begin{aligned} &-\int_{\sqrt{k}}^1 x^4 - (1+k)x^2 + k \, dx \\ &= \Big[\tfrac{1}{5}x^5 - \tfrac{1}{3}(1+k)x^3 + kx\Big]_1^{\sqrt{k}} \\ &= \big(-\tfrac{2}{15}k^{\frac{5}{2}} + \tfrac{2}{3}k^{\frac{3}{2}}\big) - \big(\tfrac{2}{15}(5k-1)\big). \end{aligned}$$

Equating the areas, much cancels, leaving

$$\begin{aligned} \tfrac{2}{15}(5k - 1) &= 0 \\ \Longrightarrow k &= \tfrac{1}{5}, \text{ as required.} \end{aligned}$$

4528. If it can be done, then $n \geq p + q$. There are p white, q black and $n - p - q$ empty squares. Take each of these as distinguishable, and we have n objects. Consider a list of the $n!$ orders of these objects. In this list, due to the fact that the counters and empty squares are not, in fact, distinguishable, we have overcounted by factors

① $p!$ for rearrangements of the white counters amongst themselves,

② $q!$ for rearrangements of the black counters amongst themselves,

③ $(n - p - q)!$ for rearrangements of the empty squares amongst themselves.

This gives $\dfrac{n!}{p! \times q! \times (n - p - q)!}$, as required.

4529. Putting the RHS over a common denominator, its numerator is $A(x-5)+B(x-3)$, which is linear. It is impossible for this to be identical to x^2+1. The student has missed the step in which the improper algebraic fraction is written as a proper algebraic fraction:

$$\begin{aligned}&\frac{x^2+1}{x^2-8x+15}\\ \equiv\ &\frac{(x^2-8x+15)+8x-14}{x^2-8x+15}\\ \equiv\ &1+\frac{8x-14}{x^2-8x+15}.\end{aligned}$$

This can now be put in the form suggested.

— Nota Bene —

Written as polynomial long division, this is

$$\begin{array}{r|l} & \quad\quad\quad\quad 1 \\ x^2-8x+15) & x^2 \quad\quad +1 \\ & -x^2+8x-15 \\ \hline & \quad\quad 8x-14 \end{array}$$

4530. We can assume the cubic is monic, without loss of generality. It has equation

$$y=(x-p)(x-q)(x-r).$$

Differentiating this by the product rule,

$$\begin{aligned}\frac{dy}{dx}&=(x-q)(x-r)+(x-p)(x-r)\\&\quad+(x-p)(x-q).\end{aligned}$$

Differentiating again by the product rule,

$$\begin{aligned}\frac{d^2y}{dx^2}&=(x-r)+(x-q)+(x-r)+(x-p)\\&\quad+(x-q)+(x-p)\\&\equiv 6x-2(p+q+r).\end{aligned}$$

Setting this to zero, the point of inflection is at

$$x=\tfrac{1}{3}(p+q+r).$$

— Nota Bene —

The product rule used for the first derivative is the three-way product rule, which is

$$(uvw)'=u'vw+uv'w+uvw'.$$

This result can be proved by using the two-way product rule twice on $\big((uv)w\big)'$.

4531. The derivative is

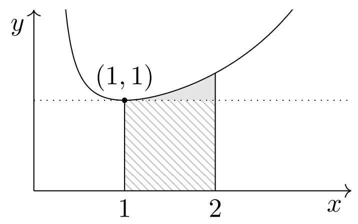
$$\frac{dy}{dx}=\frac{\sec(\ln x)\tan(\ln x)}{x}.$$

Setting this to zero for SPs,

$$\sec(\ln x)\tan(\ln x)=0.$$

Zero is not in the range of the sec function, so $\tan(\ln x)=0$. Taking the primary solution, this gives $x=1$. We are told that $x=1$ appears on the graph. Hence, the stationary point shown must have coordinates $(1,1)$.

Drawing the line $y=1$, the relevant integral (total shaded area) has a value greater than the square beneath $y=1$ (hatched):



Therefore, $\displaystyle\int_1^2 \sec(\ln x)\,dx>1$, as required.

4532. (a) Since X and Y are independent, we have 50 trials with probability of success 0.1. So, the distribution is $X+Y\sim B(50,0.1)$.

(b) The distribution in part (a) gives

$$\begin{aligned}\mathbb{P}(X+Y=2)&={}^{50}\mathrm{C}_2\times 0.1^2\times 0.9^{48}\\&=0.0779\text{ (3sf)}.\end{aligned}$$

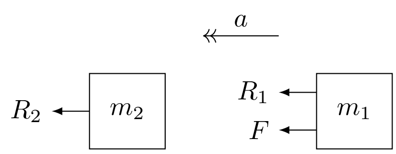
Calculating via the individual distributions, there are three possibilities for (X,Y). Their probabilities are

(x,y)	$(0,2)$	$(1,1)$	$(2,0)$
$\mathbb{P}(X=x)$	0.12156	0.27017	0.28518
$\mathbb{P}(Y=y)$	0.22766	0.14130	0.04239
p	0.02767	0.03818	0.01209

So, the combined probability is

$$\begin{aligned}\mathbb{P}(X+Y=2)&=0.02767+0.03818+0.01209\\&=0.0779\text{ (3sf)}.\end{aligned}$$

4533. The boundary case between a positive tension and a negative tension (thrust) is zero tension. In this case, the horizontal forces are as follows:



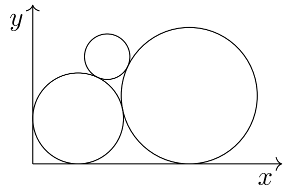
Equating the expressions for acceleration a,

$$\begin{aligned}\frac{F+R_1}{m_1}&=\frac{R_2}{m_2}\\ \Longrightarrow F+R_1&=\frac{m_1}{m_2}R_2\\ \Longrightarrow F&=\frac{m_1}{m_2}R_2-R_1.\end{aligned}$$

The force in the tow-bar will be a thrust if the braking force is larger than this, i.e. if

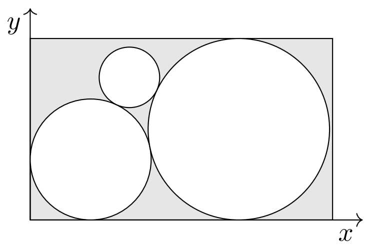
$$F>\frac{m_1}{m_2}R_2-R_1.$$

4534. The centres of the circles form a $(3, 4, 5)$ triangle. Rotating the picture so that the larger circles are tangential to the x axis, the scenario is



If the centres of the larger circles were at the same y coordinate, then their combined width would be 10 units. As it is, their width is a little less than 10 units. Likewise, if the centres of the smaller circles were at the same x coordinate, then their combined height would be 6 units. As it is, their height is a little less than 6 units. Therefore, the arrangement of circle fits, as shown below, inside a rectangle with vertices at

$$(0,0), \quad (10,0), \quad (10,6), \quad (0,6).$$



4535. The sum of the first n integers is $\frac{1}{2}n(n+1)$. We square this, giving $\frac{1}{4}n^2(n+1)^2$. So, the LHS is

$$\begin{aligned} &\tfrac{1}{4}n^2(n+1)^2 + (n+1)^3 \\ \equiv\; &(n+1)^2\big(\tfrac{1}{4}n^2 + n + 1\big) \\ \equiv\; &\tfrac{1}{4}(n+1)^2\big(n^2 + 4n + 4\big) \\ \equiv\; &\tfrac{1}{4}(n+1)^2(n+2)^2. \end{aligned}$$

This is the square of the sum of the first $n+1$ integers, as required.

——— Nota Bene ———

The above result is the *inductive step* in a proof of the following result: the sum of the first n cubes is the square of the sum of the first n integers.

4536. Consider $(1+1)^n$. Using the binomial expansion, this is given by

$$^n\mathrm{C}_0 1^n + {}^n\mathrm{C}_1 1^n + ... + {}^n\mathrm{C}_n 1^n \equiv \sum_{r=0}^{n} {}^n\mathrm{C}_r.$$

It is also equal to 2^n. QED.

4537. (a) Differentiating, the downwards speed is

$$\begin{aligned} \dot{d} &= \frac{2t(10t+25) - 10t^2}{(10t+25)^2} \\ &\equiv \frac{10t(t+5)}{(10t+25)^2}. \end{aligned}$$

At $t = 0$, both d and $\dot{d}$ are zero, so the ball is released from rest at the surface.

(b) Writing the formula as a proper fraction,

$$\begin{aligned} d &= \frac{\frac{1}{10}t(10t+25) - \frac{1}{4}(10t+25) + \frac{25}{4}}{10t+25} \\ &\equiv \tfrac{1}{10}t - \tfrac{1}{4} + \frac{5}{8t+20}. \end{aligned}$$

As $t \to \infty$, we get asymptotic approach to $d = \frac{1}{10}t - \frac{1}{4}$. So, $a = \frac{1}{10}$ and $b = \frac{1}{4}$.

4538. (a) For small-angles, $\sin\theta \approx \theta$. In the limit, this holds exactly:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

So, as the input approaches $\theta = 0$, the value of the function approaches 1. Since $\mathrm{f}(0) = 1$, the function is continuous.

(b) The maximum value is 1. The minimum value occurs at the first positive stationary point. Setting the derivative to zero,

$$\begin{aligned} &\frac{x\cos x - \sin x}{x^2} = 0 \\ \Longrightarrow\; &x\cos x - \sin x = 0. \end{aligned}$$

This is not analytically solvable. The Newton-Raphson iteration is

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n\cos x_n - \sin x_n}{-x_n \sin x_n} \\ &\equiv x_n + \frac{x_n\cos x_n - \sin x_n}{x_n \sin x_n}. \end{aligned}$$

Running this with $x_0 = 4$, $x_1 = 4.613...$ and then $x_n \to 4.49341....$ Evaluating,

$$\mathrm{sinc}(4.49341) = -0.21723....$$

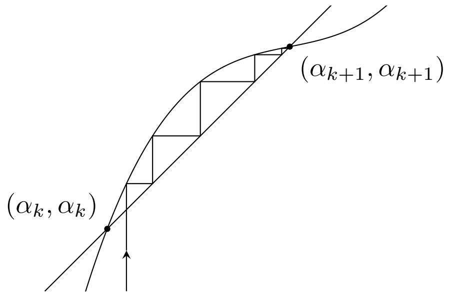
This gives the range, to 3sf, as $[-0.217, 1]$.

——— Nota Bene ———

The choice of $x_0 = 4$ for this iteration was made by considering the roots of $\mathrm{sinc}\, x$, which are at $x = n\pi$. The SP required lies between the roots at $x = \pi$ and $x = 2\pi$. So, a starting point $4 \in (\pi, 2\pi)$ was chosen.

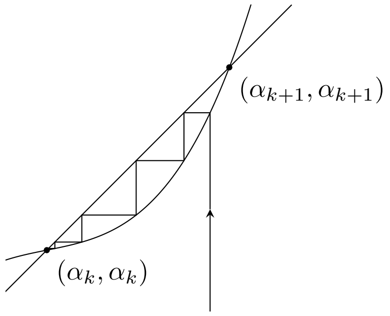
4539. Consider the graphs $y = \mathrm{g}(x)$ and $y = x$. These intersect at $x = \alpha_1, \alpha_2, ...$ So, in the interval (α_k, α_{k+1}) between two roots, the graphs do not intersect. Hence, since g is continuous, $y = \mathrm{g}(x)$ is either above $y = x$ over the entire interval, or below it. Consider these case by case:

① If $\mathrm{g}(x) > x$ on (α_k, α_{k+1}), then the behaviour is as shown. Note that g is increasing, so there are no turning points: the range is the same set as the domain.



Enacting the iteration, every starting point x_0 in the interval converges to α_{k+1}. The staircase walks up and to the right.

② If, on the other hand, $\mathrm{g}(x) < x$ on (α_k, α_{k+1}), then we have



Every starting point x_0 converges to α_k. The staircase walks down and to the left.

4540. Consider horizontals, verticals and diagonals:

- Horizontally, there are n rows to choose from. Each row gives ${}^n\mathrm{C}_{n-1} = n$ options. This gives n^2 options.
- Vertically, there are also n^2 options.
- Diagonally, there are
  - four diagonals of length $n-1$, offering a total of 4 options,
  - two diagonals of length n, offering a total of $2 \times {}^n\mathrm{C}_{n-1} = 2n$ options.

Altogether, this gives $2n^2 + 2n + 4$, as required.

4541. Multiplying by $\sin x \cos x$ and squaring, noting the possibility of having introducing new solutions,

$$\begin{aligned} &\cos x + \sin x = \sqrt{8} \sin x \cos x \\ \Longrightarrow\ &\cos^2 x + 2\sin x \cos x + \sin^2 x = 8\sin^2 x \cos^2 x. \end{aligned}$$

Using two identities,

$$\begin{aligned} &8\sin^2 x \cos^2 x - 2\sin x \cos x - 1 = 0 \\ \Longrightarrow\ &2\sin^2 2x - \sin 2x - 1 = 0. \end{aligned}$$

This is a quadratic in $\sin 2x$.

$$\begin{aligned} &(2\sin 2x + 1)(\sin 2x - 1) = 0 \\ \Longrightarrow\ &\sin 2x = -\tfrac{1}{2}, 1 \\ \Longrightarrow\ &2x = \tfrac{\pi}{2}, \tfrac{7\pi}{6}, \tfrac{11\pi}{6}, ... \\ \Longrightarrow\ &x = \tfrac{\pi}{4}, \tfrac{7\pi}{12}, \tfrac{11\pi}{12}, ... \end{aligned}$$

Testing these, $x = {}^{7\pi}/_{12}$ doesn't satisfy the original equation. So, the solution for $x \in [0, \pi)$ is

$$x = \tfrac{\pi}{4}, \tfrac{11\pi}{12}.$$

4542. The radius is given by the greatest horizontal range. We choose an arbitrary x direction and set up the (x, y) equation of the trajectory. This is

$$y = h + x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta).$$

Setting $y = 0$ and $x = r$, we rearrange to form a quadratic in $\tan\theta$:

$$\frac{gr^2}{2u^2}\tan^2\theta - r\tan\theta + \frac{gr^2}{2u^2} - h = 0.$$

For the maximum range, this equation must have exactly one root. Setting $\Delta = 0$,

$$r^2 - 4\frac{gr^2}{2u^2}\left(\frac{gr^2}{2u^2} - h\right) = 0.$$

Dividing through by r^2,

$$\begin{aligned} &1 - 4\frac{g}{2u^2}\left(\frac{gr^2}{2u^2} - h\right) = 0 \\ \Longrightarrow\ &\frac{gr^2}{2u^2} - h = \frac{u^2}{2g} \\ \Longrightarrow\ &\frac{gr^2}{2u^2} = \frac{u^2 + 2gh}{2g} \\ \Longrightarrow\ &r^2 = \frac{u^2(u^2 + 2gh)}{g^2}. \end{aligned}$$

Taking the positive square root,

$$r = \frac{u\sqrt{u^2 + 2gh}}{g}, \text{ as required.}$$

4543. Notate the moves L, R and N for left, right and no move. There are various possibilities:

Moves	Calculation	Probability
LLLL	${\frac{1}{4}}^4$	$\frac{1}{256}$
RRRR	${\frac{1}{4}}^4$	$\frac{1}{256}$
LLRR	${}^4\mathrm{C}_2 \times {\frac{1}{4}}^4$	$\frac{3}{128}$
LRNN	$\frac{4!}{2!} \times {\frac{1}{4}}^2 \times {\frac{1}{2}}^2$	$\frac{3}{16}$
NNNN	${\frac{1}{2}}^4$	$\frac{1}{16}$

The total probability is

$$p = \tfrac{1}{256} + \tfrac{1}{256} + \tfrac{3}{128} + \tfrac{3}{16} + \tfrac{1}{16} = \tfrac{9}{32}.$$

4544. The baubles must be tangent to each other, so the distance between their centres is $2r$. This is the same as the distance to the point from which they are hung. So, the strings are angled as the edges of a regular tetrahedron.

In a regular unit tetrahedron with horizontal base, the triangle formed by the apex, another vertex and the centre of the base is right-angled, with sides

$$\left(\tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{6}}{3}, 1\right).$$

So, resolving vertically for one bauble,

$$\begin{aligned} T\tfrac{\sqrt{6}}{3} &= mg \\ \Longrightarrow T &= \tfrac{\sqrt{6}}{2}mg, \text{ as required.} \end{aligned}$$

4545. Since the LHS is monic, and has an integer triple root at $x = a$, it must factorise as

$$x^4 - 7x^3 + 18x^2 - 20x + 8 \equiv (x-a)^3(x-b).$$

Equating coefficients of x^3 gives

$$3a + b = 7.$$

Hence, since a is an integer, b must also be an integer. Equating the constant terms,

$$a^3b = 8.$$

Since $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}$, a can only be 1 or 2. The (a, b) possibilities are therefore $(1, 8)$ and $(2, 1)$. The former doesn't produce the right values for the other coefficients. The latter does. Hence, the solution is $x = 1, 2$.

4546. (a) Consider an arbitrarily large value of y. If n is odd, then the equation $x^n + y^n = 1$ can be considered as a polynomial equation in x, of degree n. A polynomial of odd degree always has a real root. So, for y of any size, there is a point (x, y) on C_n. Hence, C_n is unbounded.

(b) If n is even, then x^n and y^n are both non-negative. If $y^n > 1$, then $x^n + y^n = 1$ requires $x^n < 0$. This is not possible. Hence, there are no (x, y) points on C_n for $|x| > 1$. The same argument applies for y. Hence, C_n is bounded in the square $|x| \leq 1$, $|y| \leq 1$.

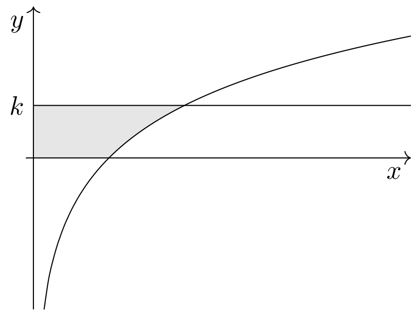
4547. Write $\cos 3\theta \equiv \cos(2\theta + \theta)$. A compound-angle formula gives

$$\cos 3\theta \equiv \cos 2\theta \cos\theta - \sin 2\theta \sin\theta.$$

Using double-angle formulae,

$$\begin{aligned} \cos 3\theta &\equiv (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ &\equiv 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta) \\ &\equiv 4\cos^3\theta - 3\cos\theta. \end{aligned}$$

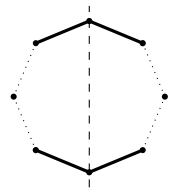
4548. The relevant area is



We integrate with respect to y:

$$\begin{aligned} \int_0^k x\,dy &= e^2 - 1 \\ \Longrightarrow \int_0^k e^y\,dy &= e^2 - 1 \\ \Longrightarrow \Big[e^y\Big]_0^k &= e^2 - 1 \\ \Longrightarrow e^k - 1 &= e^2 - 1 \\ \Longrightarrow k &= 2. \end{aligned}$$

4549. There are ${}^8\mathrm{C}_4 = 70$ ways of selecting the edges. Successful outcomes have two pairs of opposites. Exactly one of each pair of opposites must lie to the left of the line shown.



Hence, successful outcomes require the choice of two edges from the left-hand four. This gives ${}^4\mathrm{C}_2 = 6$ successful outcomes. So, the probability is $\frac{6}{70}$, which is $\frac{3}{35}$.

4550. The latter is $2\cos x = \sqrt{2}\sin y$, so $2\cos^2 x = \sin^2 y$. Using the first Pythagorean identity,

$$\begin{aligned} 2 - 2\sin^2 x &= 1 - \cos^2 y \\ \Longrightarrow \cos^2 y &= 2\sin^2 x - 1. \end{aligned}$$

Let $\sin x = s$ and $\cos y = c$. Rearranging the first equation, we have

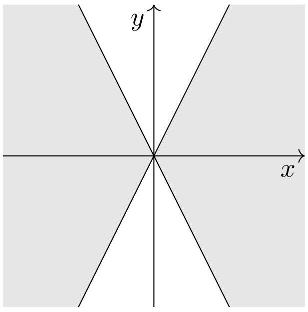
$$c = 3 + \tfrac{\sqrt{2}}{2} - 2\sqrt{3}s.$$

Substituting this into the second,

$$\begin{aligned} &\left(3 + \tfrac{\sqrt{2}}{2} - 2\sqrt{3}s\right)^2 = 2s^2 - 1 \\ \Longrightarrow\ &10s^2 - \left(12\sqrt{3} + 2\sqrt{6}\right)s + \tfrac{21}{2} + 3\sqrt{2} = 0 \\ \Longrightarrow\ &s = 0.866025..., 1.70233... \end{aligned}$$

We reject the latter root, as it is bigger than 1. The former is $s = \sqrt{3}/2$, which gives $c = \sqrt{2}/2$. Hence, the relevant solution point is $\left(\pi/3, \pi/4\right)$.

4551. The boundary equation is $4x^2 = y^2$, which is $2x = \pm y$. This is a pair of lines intersecting at the origin. Factorising to $(2x+y)(2x-y) \geq 0$, the inequality is satisfied when both factors are non-negative, or when both factors are non-positive:

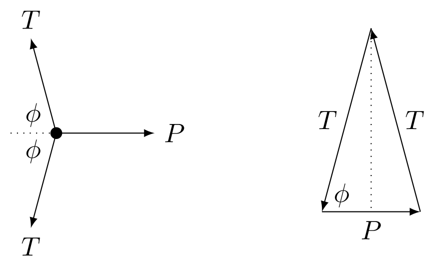


4552. This is a cubic in 3^x. Let $z = 3^x$.

$$\begin{aligned} &z^3 - 9z^2 + z - 9 = 0 \\ \Longrightarrow\ &(z-9)(z^2+1) = 0. \end{aligned}$$

The latter factor has no real roots. So, $3^x = 9$, which gives $x = 2$.

4553. The rope is smooth, so the transverse force cannot apply any friction to it. Hence, the force diagram must have a line of symmetry, i.e. the triangle of forces must be isosceles. Let $\phi = \frac{1}{2}\theta$.



Resolving horizontally, $2T\cos\phi = P$. Since ϕ is acute, we can take the positive square root in the double-angle formula $\cos^2\phi \equiv \frac{1}{2}(\cos 2\phi + 1)$, giving

$$\begin{aligned} \cos\phi &= \sqrt{\tfrac{1}{2}(1+\cos 2\phi)} \\ &= \sqrt{\tfrac{1}{2}(1+\cos\theta)}. \end{aligned}$$

Substituting this in,

$$\begin{aligned} T &= \frac{P}{2\sqrt{\frac{1}{2}(1+\cos\theta)}} \\ &\equiv \frac{P}{\sqrt{2+2\cos\theta}}, \text{ as required.} \end{aligned}$$

4554. A cubic graph has rotational symmetry around its point of inflection. If a cubic has two distinct SPs, then these, being the only points with gradient zero, must be images of one another under rotation by 180° around the point of inflection. Hence, they and the point of inflection must be collinear. □

4555. For the x intercepts, $x(e^{2-x} - 1) = 0$, which gives $x = 0, 2$. So, the area is given by

$$A = \int_0^2 x(e^{2-x} - 1)\,dx.$$

We integrate by parts. Let $u = x$ and $v' = e^{2-x} - 1$, so that $u = 1$ and $v = -e^{2-x} - x$. The indefinite integral is

$$\begin{aligned} &x(-e^{2-x} - x) + \int e^{2-x} + x\,dx \\ =\ &x(-e^{2-x} - x) - e^{2-x} + \tfrac{1}{2}x^2 + c. \end{aligned}$$

So, the area is

$$\begin{aligned} A &= \Big[-(x+1)e^{2-x} - \tfrac{1}{2}x^2\Big]_0^2 \\ &= (-5) - (-e^2) \\ &= e^2 - 5. \end{aligned}$$

4556. (a) The rate of change of z is

$$\begin{aligned} \tfrac{dz}{dt} &= \tfrac{dz}{dy} \cdot \tfrac{dy}{dx} \cdot \tfrac{dx}{dt} \\ &= -\tfrac{1}{2} \cdot 5 \cdot 4 \\ &= -10. \end{aligned}$$

(b) The rate of change of y is

$$\begin{aligned} \tfrac{dy}{dt} &= \tfrac{dy}{dx} \cdot \tfrac{dx}{dt} \\ &= 5 \cdot 4 \\ &= 20. \end{aligned}$$

This gives

$$\begin{aligned} \tfrac{d}{dt}(x+y+z) &\equiv \tfrac{dx}{dt} + \tfrac{dy}{dt} + \tfrac{dz}{dt} \\ &= 4 + 20 - 10 \\ &= 14. \end{aligned}$$

Integrating with respect to time,

$$x + y + z = 14t + c.$$

Setting each variable to 100 initially gives

$$x + y + z = 14t + 300.$$

4557. The factorial definition of ${}^n\mathrm{C}_r$ is

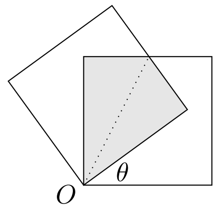
$${}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}.$$

The numerator $n!$ has a factor of n. The question, then, is whether this factor is cancelled by any in the denominator. Since n is prime, its only factors are 1 and n. So, such a cancellation can only occur if there is a factor of n in the denominator. This is only the case if $r = 0$ or $r = n$. In these cases,

$${}^n\mathrm{C}_0 = {}^n\mathrm{C}_n = 1.$$

So, the first and last entries are 1, and all of the others are multiples of n. QED.

4558. (a) The squares have a line of symmetry:

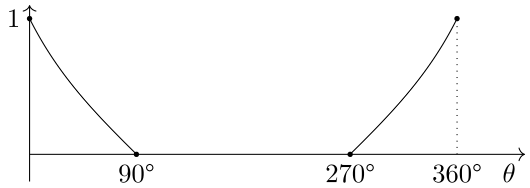


The angle subtended at O by the shaded kite is $90° - \theta$. Half of this is $45° - \frac{1}{2}\theta$. So, the kite has side lengths 1 and

$$\begin{aligned}&\tan\left(45° - \tfrac{1}{2}\theta\right)\\ \equiv\ &\frac{\tan 45° - \tan\frac{1}{2}\theta}{1 + \tan 45° \tan\frac{1}{2}\theta}\\ \equiv\ &\frac{1 - \tan\frac{1}{2}\theta}{1 + \tan\frac{1}{2}\theta}.\end{aligned}$$

Since the kite consists of two congruent right-angled triangles, each with base 1, its area is given by the above.

(b) At $\theta = 0°$, the area is 1. As θ increases, this is reduced to zero at $\theta = 90°$. The area is 0 for $\theta \in [90°, 270°]$: the squares don't overlap. Then, for $\theta \in [270°, 360°]$, the curve is the same as in $[0°, 90°]$, but this time reversed (reflected in $\theta = 180°$.

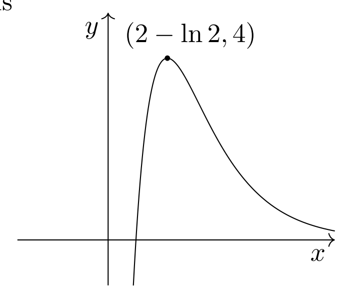


4559. Since the curve is defined for all real x, a can take any real value: $a \in \mathbb{R}$.

As $x \to -\infty$, $y \to -\infty$. And as $x \to \infty$, $y \to 0$. Looking for SPs,

$$\begin{aligned}&-4e^{2-x} + 2e^{4-2x} = 0\\ \Longrightarrow\ &2e^{2-x}(-2 + e^{2-x}) = 0\\ \Longrightarrow\ &e^{2-x} = 2\\ \Longrightarrow\ &x = 2 - \ln 2.\end{aligned}$$

This gives a stationary point at $(2 - \ln 2, 4)$. The second derivative is -8, so the point is a maximum. Curve C is



The global maximum of the curve is $y = 4$. Hence, the greatest possible value of b is 5: $b \in (-\infty, 5]$.

4560. This is the partial sum of a GP, with $a = x_n$, $r = x_n$, and $2k$ terms. Using the formula,

$$x_{n+1} = \frac{x_n\left(1 - x_n^{2k}\right)}{1 - x_n}.$$

For fixed points,

$$\begin{aligned}&x = \frac{x\left(1 - x^{2k}\right)}{1 - x}\\ \Longrightarrow\ &x - x^2 = x - x^{2k+1}\\ \Longrightarrow\ &x^{2k+1} - x^2 = 0\\ \Longrightarrow\ &x^2(x^{2k-1} - 1) = 0\\ \Longrightarrow\ &x = 0 \text{ or } x^{2k-1} = 1.\end{aligned}$$

Since $2k-1$ is odd, the latter equation gives $x = 1$. But at $x = 1$ the denominator $(1-x)$ is zero, so the original equation is undefined. We can verify that $x_n = 1$ is not a fixed point: the iteration produces $x_{n+1} = 2k > 1$.

Hence, $x = 0$ is the only fixed point, as required.

4561. Differentiating implicitly,

$$\begin{aligned}&2(x-a) + 2(y-b)\frac{dy}{dx} = 0\\ \Longrightarrow\ &\frac{dy}{dx} = -\frac{x-a}{y-b}.\end{aligned}$$

So, at (p, q), the gradient is

$$m = -\frac{p-a}{q-b}.$$

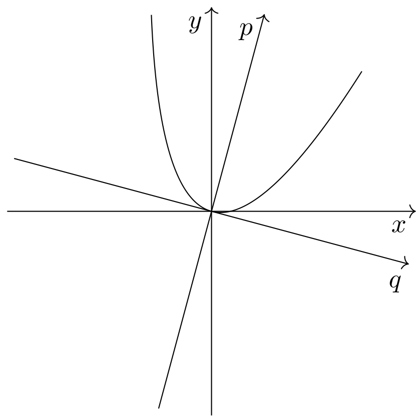
The equation of the tangent is therefore

$$\begin{aligned}&y - q = -\frac{p-a}{q-b}(x-p)\\ \Longrightarrow\ &(y-q)(q-b) = -(x-p)(p-a)\\ \Longrightarrow\ &(x-p)(p-a) + (y-q)(q-b) = 0.\end{aligned}$$

4562. Factorising the RHS, the equation is

$$ax + by = (bx - ay)^2.$$

The new variables $p = ax + by$ and $q = bx - ay$ define perpendicular coordinates axes. Variable p increases in the direction of the line $bx - ay = 0$ and variable q increases in the direction of the line $ax + by = 0$. The equation is $p = q^2$. So, the curve is a rotated parabola:



4563. (a) $\mathbb{R} \setminus \{-1\}$.

(b) Composing f with itself,

$$\begin{aligned} \text{f}^2(x) &= \frac{1}{\frac{1}{x+1}+1} \\ &\equiv \frac{x+1}{x+2}. \end{aligned}$$

So, we exclude $x = -2$ from the domain of $\text{f}^2(x)$. However, we must also exclude $x = -1$, since f is undefined there. Hence, the largest real domain over which $\text{f}^2(x)$ can be defined is $\mathbb{R} \setminus \{-1, -2\}$.

— Nota Bene —

This question shows why care has to be taken in the use of identity signs. An identity sign doesn't claim that two expressions are *always* equal, but rather that they are equal *whenever they are both defined.*

For example, $\ln x^2 \equiv 2 \ln x$ is useful identity. And there's no need to go stating domains of definition every time you write something down. But you have to keep your wits about you! The identity only holds in the domain of definition of *both sides*. If $x = -1$, then the LHS is $\ln(-1)^2 = \ln 1 = 0$, while the RHS is $2 \ln(-1)$, which is undefined.

Returning to this question, it is fine to write

$$\frac{1}{\frac{1}{x+1}+1} \equiv \frac{x+1}{x+2}.$$

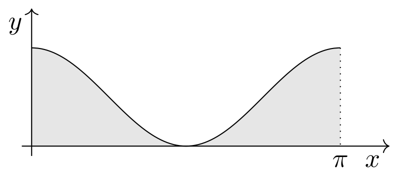
But the sides act differently at $x = 1$.

4564. Since the events are independent, we can multiply probabilities:

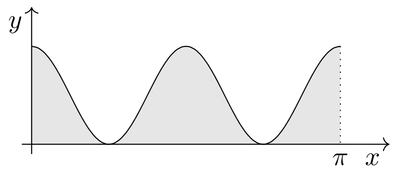
$$\begin{aligned} &\mathbb{P}(\text{at least two occurring}) \\ =\ &\mathbb{P}(\text{two or three occurring}) \\ =\ &\left(\tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \tfrac{7}{8} + \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{1}{8} + \tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \tfrac{1}{8}\right) + \left(\tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \tfrac{1}{8}\right) \\ =\ &\tfrac{3}{16}. \end{aligned}$$

4565. Consider each integral graphically:

LHS



RHS

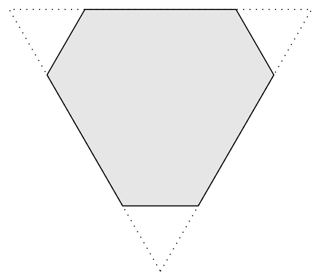


The second diagram consists of two copies of the first, with x lengths scaled by factor $\frac{1}{2}$. Hence, the areas and integrals are the same.

4566. (a) The cube has rotational symmetry order 3 around the space diagonal. So, since the hexagon is normal to the space diagonal, it must also have rotational symmetry order 3. Hence, its side lengths must be (a, b, a, b, a, b), for some $a, b \in \mathbb{R}$.

(b) Let x be the distance of each vertex of the hexagon from the nearest vertex of the cube. The side lengths are then $\sqrt{2}x$ and $\sqrt{2}(1 - x)$.

(c) The hexagon is as follows:



The large triangle has side length $\sqrt{2}(1 + x)$, thus area

$$\tfrac{\sqrt{3}}{2}(1 + x)^2$$

Each smaller triangle removed has area

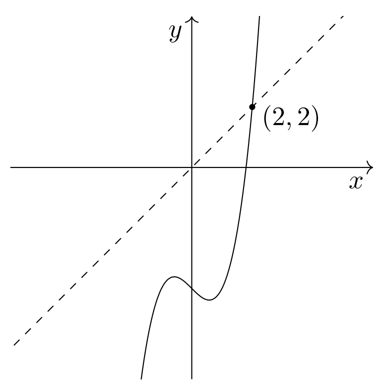
$$\tfrac{\sqrt{3}}{2}x^2.$$

So, the hexagon has area

$$A = \tfrac{\sqrt{3}}{2}\big(1 + 2x - 2x^2\big).$$

For optimisation, we set the derivative dA/dx to zero. This gives $2 - 4x = 0$, so $x = 1/2$. Each of the side lengths of the hexagon are therefore $\sqrt{2}/2$, which makes it regular. □

4567. The graph of $y = \text{g}(x) = x^3 - x - 4$, together with the line $y = x$, is as follows. Solving $x^3 - x - 4 = x$, there is a single point of intersection at $(2, 2)$.



The graph $y = \text{g}(x)$ is below $y = x$ for $x \in (-\infty, 2)$ and above $y = x$ for $x \in (2, \infty)$. Hence,

- for $x_0 < 2$, $x_n \to -\infty$,
- for $x_0 = 2$, $x_n = 2$,
- for $x_0 > 2$, $x_n \to \infty$.

4568. Separating the variables,

$$\begin{aligned} \frac{dy}{dx} &= x\cos x \operatorname{cosec} y \\ \Longrightarrow \int \sin y\,dy &= \int x\cos x\,dx. \end{aligned}$$

On the RHS, we integrate by parts, with $u = x$, $v' = \cos x$, $u' = 1$, $v = \sin x$. This gives

$$\begin{aligned} -\cos y + c &= x\sin x + \cos x \\ \Longrightarrow x\sin x + \cos x + \cos y &= c. \end{aligned}$$

This is the required result.

4569. Completing the square, $x^2 + 2x + 2 \equiv (x+1)^2 + 1$. So, let $x + 1 = \tan\theta$. This gives $dx = \sec^2\theta\,d\theta$. Enacting the substitution, then using the second Pythagorean trig identity,

$$\begin{aligned} &\int \frac{1}{(x+1)^2+1}\,dx \\ &= \int \frac{1}{\tan^2\theta + 1}\sec^2\theta\,d\theta \\ &= \int 1\,d\theta \\ &= \theta + c \\ &= \arctan(x+1) + c. \end{aligned}$$

4570. The position is $x = \sin t$, $y = 4.9$. So, the initial velocity of the projectile motion is $\dot{x} = \cos t$, $\dot{y} = 0$. Vertically, $4.9 = \frac{1}{2}gt^2$, so the particle spends 1 s in the air after the field is switched off. Horizontally, $x = x_0 + ut$. In terms of time t at which the field is switched off,

$$\begin{aligned} x &= \sin t + \cos t \\ &\equiv \sqrt{2}\sin\left(t + \tfrac{\pi}{4}\right). \end{aligned}$$

So, at landing, $x \in \left[-\sqrt{2}, \sqrt{2}\right]$.

4571. (a) The probability that the choices are the same is $\frac{1}{3}$, so the probability that the first three rounds are drawn is $\frac{1}{27}$.

(b) At least five rounds are required if the first four rounds produce at most one win.

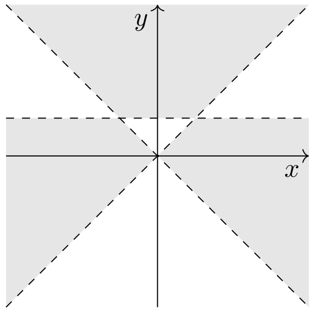
The possibilities are four draws, or three draws and one win/loss. We model the number of wins/losses with $B(4, 2/3)$:

$$\begin{aligned} p &= \tfrac{1}{3}^4 + {}^4\mathrm{C}_1 \cdot \tfrac{2}{3}^1 \cdot \tfrac{1}{3}^3 \\ &= \tfrac{1}{9}. \end{aligned}$$

4572. Factorising, the inequality is

$$\begin{aligned} -x^2y + x^2 + y^3 - y^2 &> 0 \\ \Longleftrightarrow (y-x)(y+x)(y-1) &> 0. \end{aligned}$$

The boundary equations are the lines $y = x$, $y = -x$ and $y = 1$. For the LHS to be positive, we require all three factors to be positive, or else exactly one to be positive. This gives



4573. (a) Setting $x = 0$, we get $t = 0, \pm 1$. These values also give $y = 0$. So, the curve is at the origin at $t = 0, \pm 1$.

(b) The parametric area formula is

$$A_{\text{signed}} = \int_{t_1}^{t_2} y\frac{dx}{dt}dt.$$

Substituting in the values and functions,

$$\begin{aligned} A &= -\int_0^1 (t^4 - t^2)(3t^2 - 1)\,dt \\ &= -\int_0^1 3t^6 - 4t^4 + t^2\,dt \\ &= -\left[\tfrac{3}{7}t^7 - \tfrac{4}{5}t^5 + \tfrac{1}{3}t^3\right]_0^1 dt \\ &= -\left(\tfrac{3}{7} - \tfrac{4}{5} + \tfrac{1}{3}\right) \\ &= \tfrac{4}{105}, \text{ as required.} \end{aligned}$$

— Nota Bene —

The minus sign in front of the integral is due to the fact that the loop is below the x axis. The t direction around the loop is also relevant, but in this case it maintains the sign. In practical terms, it is perfectly sensible to evaluate the integral as given by the formula and then see, after the event, whether it comes out positive or negative.

4574. The product of four consecutive integers is

$$n(n+1)(n+2)(n+3) \equiv n^4 + 6n^3 + 11n^2 + 6n.$$

Adding one to this, we want a perfect square. The expression to be squared must be a quadratic. In it, the coefficients of n^2 and n^0 must clearly be 1. So, we set up

$$n^4 + 6n^3 + 11n^2 + 6n + 1 \equiv (n^2 + an + 1)^2.$$

Expanding and comparing coefficients, $a = 3$. Summarising this,

$$n(n+1)(n+2)(n+3) \equiv (n^2 + 3n + 1)^2 - 1.$$

This proves the result by construction.

4575. (a) Yes. All that has changed is the single root at $x = 0$ (factor of x) has become a double root at $x = 0$ (factor of x^2).

(b) No. Let $z = x^2$, so that $pz^5 + qz^3 + rz = 0$. This has exactly three z roots. If two of these are positive, then this gives five x roots.

(c) Yes. Let $z = x^3$, so that $pz^5 + qz^3 + rz = 0$. This has exactly three z roots. Every number has exactly one real cube root, so these three z roots provide exactly three x roots.

4576. (a) Substituting in, $e^x - 2e^x + e^x = 0$. So, $y = e^x$ is a solution.

(b) Differentiating by the product rule,

$$\begin{aligned} & y = \text{f}(x)e^x \\ \Longrightarrow\ & \frac{dy}{dx} = \big(\text{f}'(x) + \text{f}(x)\big)\, e^x \\ \Longrightarrow\ & \frac{d^2y}{dx^2} = \big(\text{f}''(x) + 2\,\text{f}'(x) + \text{f}(x)\big)\, e^x. \end{aligned}$$

Substituting these into the DE,

$$\begin{aligned} & \big(\text{f}''(x) + 2\,\text{f}'(x) + \text{f}(x)\big)\, e^x \\ & \quad - 2\big(\text{f}'(x) + \text{f}(x)\big)\, e^x + \text{f}(x)e^x = 0. \end{aligned}$$

This simplifies to $\text{f}''(x)e^x = 0$. Since $e^x > 0$ for all x, we must have $\text{f}''(x) = 0$.

(c) Integrating, $\text{f}'(x) = A$ and then $\text{f}(x) = Ax+B$. Hence, the general solution of the DE is

$$\begin{aligned} y &= \text{f}(x)e^x \\ &= (Ax + B)e^x, \text{ as required.} \end{aligned}$$

4577. Place the first counter wlog. Put it in the corner. This leaves an $(n-1)\times(n-1)$ square. From n^2-1 locations, the probability that the second counter is placed in this square is

$$\frac{(n-1)^2}{n^2-1}.$$

Put the second counter in the corner of the reduced square, again wlog. The probability that the third counter is placed successfully is

$$\frac{(n-2)^2}{n^2-2}.$$

This pattern continues, giving

$$p = 1 \times \frac{(n-1)^2}{n^2-1} \times \frac{(n-2)^2}{n^2-2} \times ... \times \frac{1^2}{n^2-n+1}.$$

The numerator is $\big((n-1)!\big)^2$. The denominator is

$$\frac{(n^2-1)!}{(n^2-n)!}.$$

Moving $(n^2-n)!$ to the numerator,

$$p = \frac{(n^2-n)!\big((n-1)!\big)^2}{(n^2-1)!}, \text{ as required.}$$

4578. The factorisation is

$$\begin{aligned} & 2x^2 + xy - 3xz - y^2 - 6yz - 5z^2 \\ \equiv\ & (x + y + z)(2x - y - 5z). \end{aligned}$$

4579. Differentiating with respect to x,

$$\begin{aligned} & \sin y + \cos(xy) = 1 \\ \Longrightarrow\ & \cos y \frac{dy}{dx} - \sin(xy)\left(y + x\frac{dy}{dx}\right) = 0. \end{aligned}$$

Setting $\frac{dy}{dx} = 0$,

$$\begin{aligned} & -y\sin(xy) = 0 \\ \Longrightarrow\ & y = 0 \ \text{ or } \ \sin(xy) = 0. \end{aligned}$$

The former is the x axis, so not relevant to the question. From the latter, $xy = m\pi$, for $m \in \mathbb{Z}$. This is satisfied for any $x = 0$.

Substituting $x = 0$ into the equation of the curve, $\sin y + \cos(0) = 1$, so $y = n\pi$ for $n \in \mathbb{Z}$. Hence, the curve has SPs on the y axis at $(0, n\pi)$. There are infinitely many such points, as required.

4580. There are $6^4 = 1296$ possible rolls. A successful outcome needs four different numbers chosen from $\{1, 2, 3, 4, 5, 6\}$. For each such set, there is only one increasing order, yielding ${}^6\text{C}_4 = 15$ successful outcomes. So, $p = \frac{15}{1296} = \frac{5}{432}$.

4581. The derivatives are $\dot{x} = 2\cos 2t$ and $\dot{y} = \cos t$. So, the gradient is

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\cos t}{2\cos 2t}.$$

Evaluating at $t = 0$, the positive gradient at O is

$$\frac{dy}{dx} = \frac{\cos 0}{2\cos 0} = \tfrac{1}{2}.$$

So, the angle of inclination is $\arctan\frac{1}{2}$. The curve has the x axis as a line of symmetry, so the acute angle at the origin is given by

$$2\arctan\tfrac{1}{2} = 53.1° \text{ (3sf)}.$$

4582. We integrate by inspection, using the reverse chain rule. The relevant derivative is $\frac{d}{dx}(\ln x) = \frac{1}{x}$:

$$\int \frac{1}{x(\ln x)^n}\, dx = -\frac{1}{(n-1)}(\ln x)^{n-1} + c.$$

——— Nota Bene ———

Verification of the above is by the chain rule:

$$\begin{aligned}&\frac{d}{dx}\left(-\frac{1}{n-1}(\ln x)^{-(n-1)}+c\right)\\&= -\frac{1}{n-1}\cdot -(n-1)(\ln x)^{-n}\cdot\frac{1}{x}\\&\equiv (\ln x)^{-n}\cdot\frac{1}{x}\\&\equiv \frac{1}{x(\ln x)^n},\text{ as required.}\end{aligned}$$

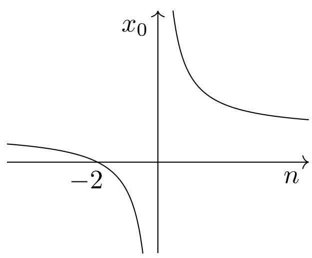
4583. Rearranging and differentiating,

$$\begin{aligned}y&=\frac{2}{1+x^n}\\ \Longrightarrow \frac{dy}{dx}&=-2(1+x^n)^{-2}\cdot nx^{n-1}\\&\equiv -\frac{2nx^{n-1}}{(1+x^n)^2}.\end{aligned}$$

Substituting $x=1$, the gradient is $-n/2$. So, the equation of the tangent is $y-1=-\frac{n}{2}(x-1)$. This intercepts the x axis at $x_0=\frac{n+2}{n}$. So, we require

$$\frac{n+2}{n}\geq 0.$$

The graph of $x_0=\dfrac{n+2}{n}$ is



So, the set of values of n for which $x_0\geq 0$ is $(\infty,-2]\cup(0,\infty)$.

4584. (a) Differentiating with respect to x,

$$\begin{aligned}y&=x\frac{dy}{dx}+\mathrm{f}\left(\frac{dy}{dx}\right)\\ \Longrightarrow \frac{dy}{dx}&=\frac{dy}{dx}+x\frac{d^2y}{dx^2}+\mathrm{f}'\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}\\ \Longrightarrow 0&=\frac{d^2y}{dx^2}\left(x+\mathrm{f}'\left(\frac{dy}{dx}\right)\right).\end{aligned}$$

So, either $\dfrac{d^2y}{dx^2}=0$ or $x+\mathrm{f}'\left(\dfrac{dy}{dx}\right)=0$.

(b) Integrating the first case twice, $y=Ax+B$.

4585. For collinearity,

$$\begin{aligned}&\frac{\cos 2a}{\sin 2a}=\frac{2\sin a}{2\cos a}\\ \Longrightarrow\ &\cos a\cos 2a=\sin a\sin 2a\\ \Longrightarrow\ &\cos a(2\cos^2 a-1)=2\sin^2 a\cos a.\end{aligned}$$

So, either $\cos a=0$ or

$$\begin{aligned}&2\cos^2 a-1=2(1-\cos^2 a)\\ \Longrightarrow\ &\cos^2 a=\tfrac{3}{4}\\ \Longrightarrow\ &\cos a=\pm\tfrac{\sqrt{3}}{2}.\end{aligned}$$

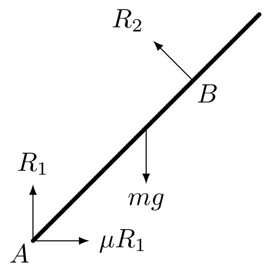
If $\cos a=0$, then $a=\frac{\pi}{2}$. At this value, the gradients are undefined. Checking the original points, they are $(0,0)$, $(0,-1)$ and $(0,2)$. These are collinear. Or, if $\cos a=\pm\sqrt{3}/2$, then $a=\frac{\pi}{6},\frac{5\pi}{6}$. So, $a\in\left\{\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6}\right\}$.

4586. The equation $\mathrm{g}\big(|x|\big)-2=0$ is

$$\begin{aligned}&\big(|x|+1\big)\big(|x|+3\big)-2=0\\ \Longrightarrow\ &|x|^2+4|x|+1=0\\ \Longrightarrow\ &|x|=-2\pm\sqrt{3}.\end{aligned}$$

Both values $-2\pm\sqrt{3}$ are negative. So, $|x|$ cannot equal either. Hence, $\mathrm{g}\big(|x|\big)-2=0$ has no roots.

4587. Consider limiting equilibrium, with friction at $F_{\max}=\mu R$. The force diagram for the ladder is



The distance AB is $\sqrt{2}l$. The equations are

$$\begin{array}{c|l}\updownarrow & R_1+R_2\frac{\sqrt{2}}{2}-mg=0,\\ \leftrightarrow & \mu R_1-R_2\frac{\sqrt{2}}{2}=0,\\ \overset{\frown}{A} & R_2\sqrt{2}l-mgl\frac{\sqrt{2}}{2}=0.\end{array}$$

Moments give $R_2=\frac{1}{2}mg$. Using the vertical,

$$\begin{aligned}&R_1+\tfrac{\sqrt{2}}{4}-mg=0\\ \Longrightarrow\ &R_1=\big(1-\tfrac{\sqrt{2}}{4}\big)mg\\ &\equiv\tfrac{4-\sqrt{2}}{4}mg.\end{aligned}$$

And the horizontal:

$$\begin{aligned}&\tfrac{4-\sqrt{2}}{4}\mu mg=\tfrac{\sqrt{2}}{4}mg\\ \Longrightarrow\ &\mu=\frac{\frac{\sqrt{2}}{4}}{\frac{4-\sqrt{2}}{4}}=\frac{\sqrt{2}}{4-\sqrt{2}}=\frac{1+2\sqrt{2}}{7}.\end{aligned}$$

Hence, for equilibrium, $\mu\geq\dfrac{1+2\sqrt{2}}{7}$, as required.

4588. Multiplying top and bottom by $\sin^2 x \cos^2 x$,

$$\begin{aligned}&\frac{\tan^2 x}{\tan^2 x + 2 + \cot^2 x}\\ \equiv\ &\frac{\sin^2 x\cos^2 x\tan^2 x}{\sin^4 + 2\sin^2 x\cos^2 x + \cos^4 x}\\ \equiv\ &\frac{\sin^4 x}{\left(\sin^2 + \cos^2 x\right)^2}\\ \equiv\ &\sin^4 x, \text{ as required.}\end{aligned}$$

4589. (a) If $a = c = e$, then the three lines are parallel. If $b = d = f$ then they are all the same line and have infinitely many points in common. Otherwise, they have no points in common.

(b) Solving the first two equations simultaneously,

$$\begin{aligned}&ax + b = cx + d\\ \implies &x = \frac{d-b}{a-c}.\end{aligned}$$

So, the first two lines intersect at

$$\left(\frac{d-b}{a-c}, \frac{a(d-b)}{a-c} + b\right).$$

For all three lines to be concurrent, this point must lie on the third line:

$$\begin{aligned}&\frac{a(d-b)}{a-c} + b = \frac{e(d-b)}{a-c} + f\\ \implies &a(d-b) + b(a-c) = e(d-b) + f(a-c)\\ \implies &ad - bc = e(d-b) + f(a-c).\end{aligned}$$

4590. This is a quadratic in e^x:

$$\begin{aligned}&e^x + e^{1-x} = e + 1\\ \implies &e^{2x} - (e+1)e^x + e = 0\\ \implies &(e^x - 1)(e^x - e) = 0\\ \implies &e^x = 1, e\\ \implies &x = 0, 1.\end{aligned}$$

4591. Rearranging the boundary equation,

$$\begin{aligned}&y^2 = 2 + 2\cos x - \sin^2 x\\ \implies &y^2 = \cos^2 x + 2\cos x + 1\\ \implies &y^2 = (\cos x + 1)^2\\ \implies &y = \pm(\cos x + 1).\end{aligned}$$

So, the upper boundary is $y = \cos x + 1$. This has x intercepts at $x = \pm\pi$. Hence, the area is

$$\begin{aligned}A &= 4\int_0^{\pi} \cos x + 1\, dx\\ &= 4\Big[\sin x + x\Big]_0^{\pi}\\ &= 4\pi.\end{aligned}$$

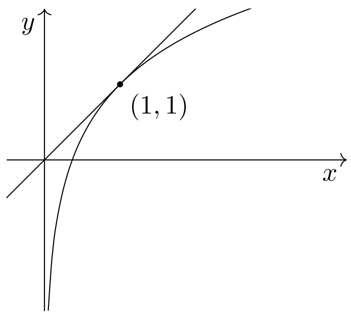
4592. Let $\sin y = x$. Differentiating implicitly,

$$\begin{aligned}&\cos y \frac{dy}{dx} = 1\\ \implies &\frac{dy}{dx} = \frac{1}{\cos y}\\ &\quad = \frac{1}{\sqrt{1 - \sin^2 y}}\\ &\quad = \frac{1}{\sqrt{1 - x^2}}, \text{ as required.}\end{aligned}$$

— Nota Bene —

In the above, we can take the positive square root in the Pythagorean identity because the domain of arcsin is $[-\pi/2, \pi/2]$. Over this domain the value of cosine is always non-negative.

4593. The domain of f is $(0, \infty)$. The boundary case, at and beyond which x_1 is non-positive, occurs when the tangent to $y = \ln x + 1$ passes through O. This tangent is at $(1, 1)$.



The behaviours are:

① If $x_0 \in (-\infty, 0]$, then x_1 is not well defined.

② If $x_0 \in (0, 1]$, then x_1 is well defined and x_2 is also well defined.

③ If $x_0 \in (1, \infty)$, then x_1 is well defined but negative, so x_2 is not well defined.

Hence, the required set is $(1, \infty)$.

4594. Simplifying the integrand,

$$\begin{aligned}&(1 - \cos t)^2\\ \equiv\ &1 - 2\cos t + \cos^2 t\\ \equiv\ &1 - 2\cos t + \tfrac{1}{2}(\cos 2t + 1)\\ \equiv\ &\tfrac{3}{2} - 2\cos t + \tfrac{1}{2}\cos 2t.\end{aligned}$$

So, the integral is

$$\begin{aligned}&\int_0^{2\pi} \tfrac{3}{2} - 2\cos t + \tfrac{1}{2}\cos 2t\, dt\\ = &\left[\tfrac{3}{2}t - 2\sin t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi}\\ = &\ 3\pi, \text{ as required.}\end{aligned}$$

4595. (a) The LHS is

$$\begin{aligned} &\mathrm{S}(n)+(n+1)^3 \\ =\ &\left(\frac{n^2+n}{2}\right)^2+(n+1)^3 \\ =\ &\frac{n^2(n+1)^2}{4}+(n+1)^3 \\ =\ &\tfrac{1}{4}(n+1)^2(n^2+4n+4) \\ =\ &\tfrac{1}{4}(n+1)^2(n+2)^2 \\ =\ &\mathrm{S}(n+1), \text{ as required.} \end{aligned}$$

(b) Since we add $(n+1)^3$, which is the $(n+1)$th cube, to $\mathrm{S}(n)$ to get $\mathrm{S}(n+1)$, the underlying sequence must be $u_n = n^3$. To verify this, we should also check $\mathrm{S}(1)$, which should be 1:

$$\mathrm{S}(1) = \left(\frac{1+1}{2}\right)^2 = 1.$$

So, $\mathrm{S}(n)$ is the sum of the first n cubes.

4596. The volume of water is $\frac{4}{3}\pi r^3$. Setting this to the volume of the cone, the height y of the cone is

$$\begin{aligned} &\tfrac{4}{3}\pi r^3 = \tfrac{1}{3}\pi r^2 y \\ \Longrightarrow\ &y = 4r. \end{aligned}$$

When the depth in the sphere is h, the spherical cap formula gives the volume of water in the sphere as $\frac{2}{3}\pi r^2 h$. This is the same volume as the empty space in the cone. The empty space in the cone is a cone itself, with height $(4r - H)$ and radius $\frac{1}{4}(4r - H)$. So, its volume is

$$\begin{aligned} &\tfrac{1}{3}\pi \cdot \left(\tfrac{1}{4}(4r-H)\right)^2(4r-H) = \tfrac{2}{3}\pi r^2 h \\ \Longrightarrow\ &(4r-H)^3 = 32r^2 h, \text{ as required.} \end{aligned}$$

4597. Factorising the boundary equation,

$$\begin{aligned} &2x^5 - 3x^3 + x = 0 \\ \Longrightarrow\ &x\big(\sqrt{2}x+1\big)\big(\sqrt{2}x-1\big)(x+1)(x-1) \\ \Longrightarrow\ &x = 0, \pm 1, \pm\tfrac{\sqrt{2}}{2}. \end{aligned}$$

Each of these is a single root. So, there is a sign change in $2x^5 - 3x^3 + x$ at each. This gives the solution as

$$x \in \left[-1, -\sqrt{2}/2\right] \cup \left[0, \sqrt{2}/2\right] \cup [1, \infty).$$

4598. The boundary equation is

$$\begin{aligned} &\sin 2t = k\cos t \\ \Longrightarrow\ &2\sin t\cos t - k\cos t = 0 \\ \Longrightarrow\ &\cos t(2\sin t - k) = 0 \\ \Longrightarrow\ &\cos t = 0 \quad\text{or}\quad \sin t = \tfrac{k}{2}. \end{aligned}$$

Both excesses repeat after $[0, 2\pi]$, so we need only consider that domain. There are always roots at $t = \frac{\pi}{2}, \frac{3\pi}{2}$. The form of the solution set of $\sin t = \frac{k}{2}$ depends on the value of k.

① For $k \geq 2$, $\sin t = \frac{k}{2}$ has no roots apart from those of $\cos t = 0$, so the solution set of $E_1 > E_2$ is $(\pi/2, 3\pi/2)$. E_1 is greater than E_2 for half the time.

② For $k \in (0, 2)$, $\sin t = \frac{k}{2}$ has distinct roots $t = \theta \in (0, \pi/2)$ and $t = \pi - \theta$. Since these are symmetrical around $t = \pi/2$, the proportion of time remains at a half.

③ For $k = 0$, the result is trivial.

④ For $k \in (-2, 0)$, $\sin t = \frac{k}{2}$ has two distinct roots $t = \theta \in (\pi, 3\pi/2)$ and $t = 3\pi - \theta$. Since these are symmetrical around $t = 3\pi/2$, the proportion of time remains at a half.

⑤ For $k \leq -2$, $\sin t = \frac{k}{2}$ has no roots apart from those of $\cos t = 0$, so the solution set of $E_1 > E_2$ is $[0, \pi/2) \cup (3\pi/2, 2\pi]$. Again, E_1 is greater than E_2 for half the time.

Hence, independent of the value of k, $E_1 > E_2$ is satisfied half of the time.

4599. (a) Solving simultaneously,

$$\begin{aligned} &x^2 + m^2(x+1)^2 = 1 \\ \Longrightarrow\ &(1+m^2)x^2 + 2m^2x + m^2 - 1 = 0 \\ \Longrightarrow\ &(x+1)\big((1+m^2)x - 1 + m^2\big) = 0 \\ \Longrightarrow\ &x = -1 \quad\text{or}\quad x = \frac{1-m^2}{1+m^2}. \end{aligned}$$

(b) The y coordinate at B is

$$\begin{aligned} y &= m\left(\frac{1-m^2}{1+m^2} + 1\right) \\ &= \frac{2m}{m^2+1}. \end{aligned}$$

The normal to L_m has gradient $-\frac{1}{m}$ and passes through

$$\left(\frac{1-m^2}{1+m^2}, \frac{2m}{m^2+1}\right).$$

So, it has equation

$$y - \frac{2m}{m^2+1} = -\frac{1}{m}\left(x - \frac{1-m^2}{1+m^2}\right).$$

(c) Substituting $x = 1$,

$$\begin{aligned} y &= \frac{2m}{m^2+1} - \frac{1}{m}\left(1 - \frac{1-m^2}{1+m^2}\right) \\ &= \frac{2m}{m^2+1} - \frac{2m}{m^2+1} \\ &= 0. \end{aligned}$$

So, the normal to L_m which passes through B also passes through the point $(1, 0)$. This sets up an angle in a semicircle, which is duly a right angle. QED.

4600. (a) Substituting $S = 62.1$ and $D = 51.3$ gives

$$\frac{dS}{dt} = -1.3432,$$
$$\frac{dD}{dt} = 2.5842.$$

(b) If these rates remain constant, then supply and demand change linearly with time. Supply is given by $S = -1.3432t + 57.1$ and demand by $D = 2.5842t + 51.3$. Equating these,

$$-1.3432t + 57.1 = 2.5842t + 51.3$$
$$\implies t = 1.477 \text{ (4sf)}.$$

(c) The first month is as in part (b). At $t = 1$,

$$S = 55.7568,$$
$$D = 53.8842.$$

Recalculating the rates,

$$\frac{dS}{dt} = -1.2547386,$$
$$\frac{dD}{dt} = 2.6257662.$$

The new linear equations are

$$S - 55.7568 = -1.2547386(t - 1),$$
$$D - 53.8842 = 2.6257662(t - 1).$$

Setting $S = D$, we solve to give $t = 1.483$ (4sf).

——— End of 46th Hundred ———